

NUMERICAL ANALYSIS

ENG 205 Formula Sheet

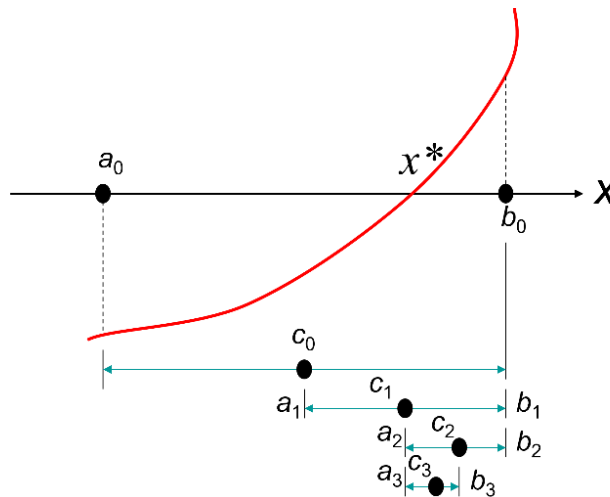
$$b_n - a_n = \frac{b_0 - a_0}{2^n}$$

$$\begin{aligned} |c_n - x^*| &\leq c_n - a_n \\ &\leq \frac{b_0 - a_0}{2^{n+1}} \end{aligned}$$

For the previous example

$$\frac{0.4}{2^{n+1}} < 10^{-4}$$

$$\boxed{n = 11}$$



Number of Iterations

$$|c_n - x^*| \leq c_n - a_n \leq \frac{b_0 - a_0}{2^{n+1}}$$

Newton Raphson Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Secant Method

$$x_n = \frac{x_{n-1}f(x_{n-2}) - x_{n-2}f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}$$

False Position

$$x_n = \frac{x_{n-1}f(x_{n-2}) - x_{n-2}f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}$$

Newton Raphson Method

(multivariable)

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} - \underline{J}^{-1}(x_1^{(k-1)}, x_2^{(k-1)}) \begin{bmatrix} f_1(x_1^{(k-1)}, x_2^{(k-1)}) \\ f_2(x_1^{(k-1)}, x_2^{(k-1)}) \end{bmatrix}$$

$$\underline{x}^{(k)} = \underline{x}^{(k-1)} - \underline{J}^{-1}(\underline{x}^{(k-1)}) \underline{f}(\underline{x}^{(k-1)})$$

ITERATIVE METHODS (I): Gauss-Jacobi Method

$$\begin{aligned} x_1^{(k)} &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)}) \\ x_2^{(k)} &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k-1)} - a_{23}x_3^{(k-1)} - \dots - a_{2n}x_n^{(k-1)}) \\ x_3^{(k)} &= \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k-1)} - a_{32}x_2^{(k-1)} - \dots - a_{3n}x_n^{(k-1)}) \\ &\vdots \\ x_n^{(k)} &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k-1)} - a_{n2}x_2^{(k-1)} - \dots - a_{n,n-1}x_{n-1}^{(k-1)}) \end{aligned}$$

ITERATIVE METHODS (I): Gauss-Seidel Method

$$\begin{aligned} x_1^{(k)} &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)}) \\ x_2^{(k)} &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)} - \dots - a_{2n}x_n^{(k-1)}) \\ x_3^{(k)} &= \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - \dots - a_{3n}x_n^{(k-1)}) \\ &\vdots \\ x_n^{(k)} &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \dots - a_{n,n-1}x_{n-1}^{(k)}) \end{aligned}$$

Convergence of Iterative Methodz; Gauss-Jacobi

$$\underline{x}(k) = \underline{D}^{-1}\underline{B} - \underline{D}^{-1}(\underline{L} + \underline{U}) \underline{x}(k-1) \quad \underline{I} = -\underline{D}^{-1}(\underline{L} + \underline{U}) \quad \underline{C} = \underline{D}^{-1}\underline{B}$$

Gauss-Seidel

$$\underline{x}^{(k)} = (\underline{L} + \underline{D})^{-1} \underline{B} - (\underline{L} + \underline{D})^{-1} \underline{U} \underline{x}^{(k-1)}$$

$$\underline{T} = -(\underline{L} + \underline{D})^{-1} \underline{U}$$

$$\underline{C} = (\underline{L} + \underline{D})^{-1} \underline{B}$$

$$\sqrt{\rho(\underline{T}^T \underline{T})} < 1$$

Optimal Relaxation Factor

Gauss-Jacobi

$$\underline{T} = \underline{I} - \omega \underline{D}^{-1} \underline{A}$$

Gauss-Seidel

$$\underline{T} = (\underline{D} + \omega \underline{L})^{-1} [(1 - \omega)\underline{D} - \omega \underline{U}]$$

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - [\rho(\underline{T}_j)]^2}}$$

Gauss-Jacobi Method

$$\underline{x}^{(k)} = \underline{D}^{-1} \underline{B} - \underline{D}^{-1} (\underline{L} + \underline{U}) \underline{x}^{(k-1)}$$

Gauss-Seidel Method

$$\underline{x}^{(k)} = (\underline{D} + \underline{L})^{-1} \underline{B} - (\underline{D} + \underline{L})^{-1} \underline{U} \underline{x}^{(k-1)}$$

Curve-Fitting

$$\begin{bmatrix} \sum_{k=1}^N 1 & \sum_{k=1}^N x_k & \sum_{k=1}^N x_k^2 & \cdots & \sum_{k=1}^N x_k^n \\ \sum_{k=1}^N x_k & \sum_{k=1}^N x_k^2 & \sum_{k=1}^N x_k^3 & \cdots & \sum_{k=1}^N x_k^{n+1} \\ \sum_{k=1}^N x_k^2 & \sum_{k=1}^N x_k^3 & \sum_{k=1}^N x_k^4 & \cdots & \sum_{k=1}^N x_k^{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^N x_k^n & \sum_{k=1}^N x_k^{n+1} & \sum_{k=1}^N x_k^{n+2} & \cdots & \sum_{k=1}^N x_k^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^N y_k \\ \sum_{k=1}^N y_k x_k \\ \sum_{k=1}^N y_k x_k^2 \\ \vdots \\ \sum_{k=1}^N y_k x_k^n \end{bmatrix}$$

LAGRANGE APPROXIMATION

$$P(x) = \sum_{i=0}^n L_i(x) y_i \quad L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^n \frac{x - x_i}{x_k - x_i}$$

INTERPOLATION AND POLYNOMIAL APPROXIMATION

Newton Polynomial

Special case:

When x_0, x_1, \dots, x_n are arranged consecutively with equal spacing $h = x_{n+1} - x_n$

| x | f | 1 st Divided differences | 2 nd Divided differences | 3 rd Divided differences | 4 th Divided differences |
|-------|-------|--|--|--|---|
| x_0 | f_0 | $\frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}$ | $\frac{\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}}{2h} = \frac{\Delta^2 f_0}{2h^2}$ | $\frac{\frac{\Delta^2 f_1}{2h^2} - \frac{\Delta^2 f_0}{2h^2}}{3h} = \frac{\Delta^3 f_0}{6h^3}$ | $\frac{\Delta^4 f_0}{24h^4} \dots\dots$ |
| x_1 | f_1 | $\frac{f_2 - f_1}{h} = \frac{\Delta f_1}{h}$ | $\frac{\frac{\Delta f_2}{h} - \frac{\Delta f_1}{h}}{2h} = \frac{\Delta^2 f_1}{2h^2}$ | \vdots | \vdots |
| x_2 | f_2 | $\frac{f_3 - f_2}{h} = \frac{\Delta f_2}{h}$ | \vdots | \vdots | \vdots |
| x_3 | f_3 | \vdots | \vdots | \vdots | \vdots |

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$